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THEORY OF AN ELASTIC–PLASTIC COSSERAT SURFACE

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Abstract—This paper is concerned with constitutive equations and related thermodynamical restrictions for an elastic–plastic Cosserat surface. The main developments are carried out in the context of the nonlinear theory. Various special cases, including that with infinitesimal deformation, are discussed. A linear theory of an isotropic, elastic–perfectly plastic Cosserat surface (and plate) is given detailed attention.

1. INTRODUCTION

RECENTLY, Green, Naghdi and Wainwright [1] have presented a general theory of a Cosserat surface, i.e., a surface embedded in a Euclidean 3-space to every point of which a single director is assigned. The approach in [1] is based upon a postulated energy balance, an entropy production inequality and invariance requirements under superposed rigid body motions. The resulting theory is completely general in the sense that it is neither restricted to infinitesimal deformation nor to elastic materials. Apart from the derivation of the general theory, a number of other results are discussed in [1]: These include the derivation of nonlinear constitutive equations for an elastic Cosserat surface and a systematic discussion of the corresponding linearized theory. This linear theory, with suitable identification of the variables involved, may be regarded as including a number of the existing classical theories of shells and plates as special cases.

The present paper is concerned with an elastic–plastic Cosserat surface, the development of which is based on the general theory of an elastic–plastic continuum given by Green and Naghdi [2].* In contrast to the existing theories of elastic–plastic plates and shells, which are usually restricted to infinitesimal deformation, our main development here is carried out for finite deformation and the infinitesimal theory is deduced as a special case. In the main development of the paper, pertaining to the construction of the constitutive equations, considerable economy of space is achieved by employing a compact notation. The basic kinematical variables of the Cosserat surface arising from the components of the displacement, the director displacement and their derivatives are collectively labeled as *generalized strain*. Similarly, the vector and tensor fields arising from the primitive force and director force (both per unit length and acting across a curve on the surface) are collectively called *generalized stress*.

* See also [3].

Following some preliminaries in Section 2, we consider nonlinear constitutive equations for an elastic–plastic Cosserat surface in Section 3. We begin our analysis by admitting an additional set of kinematic variables, collectively called *generalized plastic strain*, and assume that it has the same invariance properties as the generalized strain mentioned above. After introducing a constitutive assumption for the generalized strain (which depends on temperature, generalized stress and generalized plastic strain), we specify certain additional properties of the generalized plastic strain and proceed to develop the constitutive equations for the generalized plastic strain rate. These equations, which are independent of the time scale used in computing the rate of change, are obtained with the help of loading criteria and loading (or yield) surfaces, i.e., hypersurfaces in a thirteen dimensional Euclidean space of generalized stress and temperature. The constitutive relations of Section 3 are valid for an anisotropic elastic–plastic Cosserat surface. Using a suitable form of an entropy production inequality, thermodynamical restrictions which must hold in the presence of the generalized plastic strain are deduced in Section 4. The special cases of an elastic–perfectly plastic and a rigid–plastic Cosserat surface are briefly discussed in Sections 5 and 6, respectively.

The remainder of the paper (Sections 7–9) is concerned with the infinitesimal deformation of an elastic–perfectly plastic Cosserat surface (and plate), where the initial director is chosen to be coincident with the unit normal to the initial surface. A linear theory of an elastic–perfectly plastic Cosserat surface whose constitutive equations correspond to those of a transversely isotropic material in a three dimensional theory are discussed in Section 7. These constitutive relations are such that in the case of an initially flat Cosserat surface, i.e., a Cosserat plate (Section 8), the differential equations of the complete theory separate into those for extensional deformation and those for bending deformation. Finally, using the results of Section 8, we relate in Section 9 some of the constitutive coefficients of the isothermal linear theory to corresponding known coefficients in the three dimensional theory.

2. NOTATION AND PRELIMINARIES

We summarize here some of the basic equations of the theory of a Cosserat surface. For additional details we refer the reader to [1].

Consider a surface $\mathscr{s}$, embedded in a Euclidean 3-space, defined by

$$\mathbf{r} = \mathbf{r}(x^\alpha, t), \tag{2.1}$$

where $\mathbf{r}$ is the position vector of a point on $\mathscr{s}$ relative to a fixed reference frame, t is the time, and $x^\alpha(\alpha = 1, 2)$ are regarded as convected coordinates which define points of $\mathscr{s}$. If $\mathbf{a}_\alpha$ are the base vectors along the x^α coordinate curves, then

$$\begin{aligned} &\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}, \qquad \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\alpha\beta}, \qquad a = \det(a_{\alpha\beta}) > 0, \\ &\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta, \qquad \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta}, \qquad \mathbf{a}^\alpha = a^{\alpha\gamma}\mathbf{a}_\gamma, \end{aligned} \tag{2.2}$$

and the unit normal $\mathbf{a}_3$ to $\mathscr{s}$ may be defined by

$$\mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0, \qquad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1, \qquad \mathbf{a}^3 = \mathbf{a}_3, \qquad [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] > 0, \tag{2.3}$$

where a comma denotes partial differentiation with respect to x^α, $a_{\alpha\beta}$ is the first fundamental form of the surface, $a^{\alpha\beta}$ is its conjugate and δ^α_β is the Kronecker symbol in 2-space. Also, the second fundamental form of $\mathscr{s}$ is given by

$$b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta}. \tag{2.4}$$

Throughout this paper, Greek indices take the values 1, 2, the usual summation convention is employed, a comma stands for partial differentiation with respect to the coordinates x^α and a vertical line denotes covariant differentiation with respect to $a_{\alpha\beta}$. We make no distinction between functions and function values and, where convenient, we may omit specific reference to the independent variables of a given function and write, say, $\mathbf{r}_{,\alpha}$ for $\mathbf{r}(x^\gamma, t)_{,\alpha}$. Also, we refer to the surface at some initial time t_0 by $\mathscr{S}$. The initial surface base vectors, the initial unit normal and the initial first and second fundamental forms of $\mathscr{S}$ will be designated by $\mathbf{A}_\alpha$, $\mathbf{A}_3$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$, respectively.

A surface to every point of which a single director $\mathbf{d}$ is assigned is called a Cosserat surface. The director is not necessarily along the normal to $\mathfrak{s}$ and is assumed to have the property that it remains invariant in length when the motion of $\mathfrak{s}$ is altered only by superposed rigid body motion; the initial value of the director on $\mathscr{S}$ will be denoted by $\mathbf{D}$. As in [1], we regard the motion of a Cosserat surface to be characterized by

$$\mathbf{r} = \mathbf{r}(x^\alpha, t), \qquad \mathbf{d} = \mathbf{d}(x^\alpha, t), \qquad [\mathbf{a}_1\mathbf{a}_2\mathbf{d}] > 0. \tag{2.5}$$

In terms of its components, the director $\mathbf{d}$ can be expressed in the form

$$\mathbf{d} = d^\alpha\mathbf{a}_\alpha + d^3\mathbf{a}_3 = d_\alpha\mathbf{a}^\alpha + d_3\mathbf{a}^3, \tag{2.6}$$

and the velocity of a point of $\mathfrak{s}$ and the director velocity at $\mathbf{r}$ are given by

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{w} = \dot{\mathbf{d}}, \tag{2.7}$$

where a superposed dot denotes differentiation with respect to time holding the convected coordinates fixed. We also define the kinematical quantities

$$\begin{aligned} \lambda_{\alpha\beta} &= \mathbf{a}_\alpha \cdot \mathbf{d}_{,\beta} = d_{\alpha|\beta} - b_{\alpha\beta}d_3, \\ \lambda_{3\beta} &= \mathbf{a}_3 \cdot \mathbf{d}_{,\beta} = d_{3,\beta} + b^\alpha_\beta d_\alpha, \end{aligned} \tag{2.8}$$

where a vertical line stands for covariant differentiation with respect to $a_{\alpha\beta}$. The corresponding quantities on the initial surface $\mathscr{S}$ are defined by

$$\begin{aligned} \Lambda_{\alpha\beta} &= \mathbf{A}_\alpha \cdot \mathbf{D}_{,\beta} = D_{\alpha\|\beta} - B_{\alpha\beta}D_3, \\ \Lambda_{3\beta} &= \mathbf{A}_3 \cdot \mathbf{D}_{,\beta} = D_{3,\beta} + B^\alpha_\beta D_\alpha, \end{aligned} \tag{2.9}$$

where double vertical lines in $(2.9)_1$ denote covariant differentiation with respect to $A_{\alpha\beta}$. We now introduce the following kinematical variables, namely

$$\begin{aligned} 2e_{\alpha\beta} &= a_{\alpha\beta} - A_{\alpha\beta}, \\ \kappa_{\alpha\beta} = \lambda_{\alpha\beta} - \Lambda_{\alpha\beta}, &\qquad \kappa_{3\beta} = \lambda_{3\beta} - \Lambda_{3\beta}, \\ \delta_\alpha = d_\alpha - D_\alpha, &\qquad \delta_3 = d_3 - D_3, \end{aligned} \tag{2.10}$$

and for convenience refer to these collectively simply as "strains". We remark that the kinematical variables defined in (2.10) have the property that they remain unaltered by superposed rigid body motions.

Let a curve c be defined through some point p on the surface $\mathfrak{s}$ and let $\mathbf{v} = v^\alpha\mathbf{a}_\alpha$ be the unit normal to c lying in the surface. Further, let $\mathbf{N}$ be the (physical) force vector per unit length at p, exerted by one part of the surface on the other across c, and let $\mathbf{n}^\alpha$ represent the (physical) curve force vectors over the α-th coordinate line. Then,

$$\mathbf{N} = \mathbf{N}^\alpha v_\alpha, \qquad \mathbf{N}^\alpha = \mathbf{n}^\alpha(a^{\alpha\alpha})^{1/2}, \tag{2.11}$$

and we can express the vector $\mathbf{N}^\alpha$ in terms of its components by*

$$\mathbf{N}^\alpha = N^{\gamma\alpha}\mathbf{a}_\gamma + N^{3\alpha}\mathbf{a}_3. \tag{2.12}$$

Similarly, let $\mathbf{M}$ be the (physical) director force per unit length at p over the curve c, and let $\mathbf{m}^\alpha$ represent the (physical) director force vectors over the α-th coordinate line. Then,*

$$\mathbf{M} = \mathbf{M}^\alpha v_\alpha, \qquad \mathbf{M}^\alpha = \mathbf{m}^\alpha(a^{\alpha\alpha})^{1/2}, \tag{2.13}$$

and we can put

$$\mathbf{M}^\alpha = M^{\gamma\alpha}\mathbf{a}_\gamma + M^{3\alpha}\mathbf{a}_3. \tag{2.14}$$

Based on a postulated energy balance and the invariance requirements under superposed rigid body motions, Green *et al.* in [1] deduce the equation of the conservation of mass

$$\frac{D}{Dt}(\rho a^{1/2}) = 0, \tag{2.15}$$

and the equations of motion which have the following component form:

$$\begin{aligned} N^{\alpha\beta}|_\beta - b^\alpha_\beta N^{3\beta} + \rho F^\alpha &= \rho c^\alpha, & N^{3\beta}|_\beta + b_{\alpha\beta}N^{\beta\alpha} + \rho F^3 &= \rho c^3, \\ M^{\alpha\beta}|_\beta - b^\alpha_\beta M^{3\beta} + \rho \bar{L}^\alpha &= m^\alpha, & M^{3\beta}|_\beta + b_{\alpha\beta}M^{\beta\alpha} + \rho \bar{L}^3 &= m^3. \end{aligned} \tag{2.16}$$

In (2.15) and (2.16), ρ is the mass per unit area of $\mathscr{s}$, $\{c^\alpha, c^3\}$ are the components of acceleration, $\mathbf{F}$ with components $\{F^\alpha, F^3\}$ is the assigned surface force per unit mass, $\bar{\mathbf{L}}$ is the difference of the assigned surface director force per unit mass and the inertia terms due to the director displacement $\mathbf{d}$, and $\mathbf{m}$ with components $\{m^\alpha, m^3\}$ is a quantity which we regard as being *defined* by equations† $(2.16)_{3,4}$. In addition, the equation of moments has the component form

$$\begin{aligned} N^{3\alpha} + m^3 d^\alpha - m^\alpha d^3 + M^{3\gamma}\lambda^\alpha_{\cdot\gamma} - M^{\alpha\gamma}\lambda^3_{\cdot\gamma} &= 0, \\ N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} - m^\alpha d^\beta - M^{\alpha\gamma}\lambda^\beta_{\cdot\gamma}&. \end{aligned} \tag{2.17}$$

The reduced form of the local energy equation is

$$-\rho(\dot{A} + T\dot{S} + S\dot{T}) + N'^{\alpha\beta}\dot{e}_{\alpha\beta} + M^{\alpha\beta}\dot{\kappa}_{\alpha\beta} + M^{3\beta}\dot{\kappa}_{3\beta} + m^\alpha\dot{\delta}_\alpha + m^3\dot{\delta}_3 + \rho r - q^\alpha|_\alpha = 0, \tag{2.18}$$

and the local Clausius–Duhem inequality is given by

$$\rho T\dot{S} - \rho r + q^\alpha|_\alpha - \frac{q^\alpha T_{,\alpha}}{T} \geqq 0. \tag{2.19}$$

Here A is the Helmholtz free energy per unit mass, S is the entropy per unit mass, $T(> 0)$ is the temperature and r is the heat supply per unit mass per unit time. Also,

$$h = q^\alpha v_\alpha, \qquad q^\alpha = h^\alpha(a^{\alpha\alpha})^{1/2}, \tag{2.20}$$

* The order of the indices here is the same as that used in [1]; but is the reverse of that conventionally used in shell theory, e.g., Green and Naghdi [4]. The relation $(2.13)_1$ holds when the constitutive assumptions are such that $(\mathbf{M} - \mathbf{M}^\alpha v_\alpha)$ does not depend explicitly on the director velocity and, in particular, it is valid for elastic materials. Here we also adopt this special relation for elastic–plastic materials.

† The vector $\mathbf{m}$ arises as a natural consequence of the reduction of the general integral form of the energy balance equation (see [1], Section 3). Equations $(2.16)_{3,4}$ may be considered as director equations of motion. In general constitutive equations are required for both $\mathbf{m}$ and $\mathbf{M}^\alpha$.

where h is the flux of heat across c per unit length per unit time and h^{α} denotes the flux of heat across the x^{α}-curves on $\mathscr{s}$.

3. AN ELASTIC–PLASTIC COSSERAT SURFACE

The theory of an elastic–plastic Cosserat surface presented here closely parallels the general theory of an elastic–plastic continuum as developed by Green and Naghdi [2]. An alternative, but entirely equivalent, development of the theory of elastic–plastic continua in 3-space is included in [3]. We refer the reader to these papers for further details.

In order to simplify the development of the theory, yet still maintain a notation consistent with that in [1], we introduce the following compact notation. Let $\mathbf{\Theta}$ denote the ordered set

$$\mathbf{\Theta} = (\mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, a) \tag{3.1}$$

where $\mathbf{A}$, $\mathbf{B}$ are second order surface tensors, $\mathbf{a}$, $\mathbf{b}$ are surface vectors, and a is a scalar. $\mathbf{\Theta}$ belongs to a thirteen dimensional vector space $\mathscr{a}$; if $\mathbf{\Lambda}$ is any other element of $\mathscr{a}$, say

$$\mathbf{\Lambda} = (\mathbf{C}, \mathbf{D}, \mathbf{c}, \mathbf{d}, c),$$

then addition and scalar multiplication on $\mathscr{a}$ are defined by

$$\alpha\mathbf{\Theta} + \beta\mathbf{\Lambda} = (\alpha\mathbf{A} + \beta\mathbf{C}, \alpha\mathbf{B} + \beta\mathbf{D}, \alpha\mathbf{a} + \beta\mathbf{c}, \alpha\mathbf{b} + \beta\mathbf{d}, \alpha a + \beta c) \tag{3.2}$$

for arbitrary real scalars α, β. We define a scalar product on $\mathscr{a}$ by

$$\begin{aligned} \mathbf{\Theta} \,.\, \mathbf{\Lambda} &= \mathrm{tr}(\mathbf{A}\mathbf{C}^T) + \mathrm{tr}(\mathbf{B}\mathbf{D}^T) + \mathbf{a} \,.\, \mathbf{c} + \mathbf{b} \,.\, \mathbf{d} + ac \\ &= A_{\alpha\beta}C^{\alpha\beta} + B_{\alpha\beta}D^{\alpha\beta} + a_{\alpha}c^{\alpha} + b_{\alpha}d^{\alpha} + ac, \end{aligned} \tag{3.3}$$

where tr denotes the trace operator and the notation $\mathbf{C}^T$ stands for the transpose of $\mathbf{C}$. Linear transformations mapping $\mathscr{a}$ into itself are designated by

$$\mathbf{L}[\mathbf{\Theta}] = \mathbf{\Lambda}, \qquad \mathbf{\Theta}, \mathbf{\Lambda} \in \mathscr{a} \tag{3.4}$$

and have the component form

$$(\overset{(1)}{l}{}^{\alpha\beta\gamma\delta}A_{\gamma\delta},\ \overset{(2)}{l}{}^{\alpha\beta\gamma\delta}B_{\gamma\delta},\ \overset{(1)}{l}{}^{\alpha\beta}a_{\beta},\ \overset{(2)}{l}{}^{\alpha\beta}b_{\beta},\ la) = (C^{\alpha\beta},\ D^{\alpha\beta},\ c^{\alpha},\ d^{\alpha},\ c). \tag{3.5}$$

Let f denote a scalar-valued function on an open subset of $\mathscr{a}$; we denote the differential of f at $\mathbf{\Theta}$ by

$$\frac{\partial}{\partial\mathbf{\Theta}} f(\mathbf{\Theta}) \,.\, \mathbf{\Lambda} \quad \text{for any } \mathbf{\Lambda} \in \mathscr{a}. \tag{3.6}$$

This has the component form*

$$\frac{\partial f}{\partial A_{\alpha\beta}} C_{\alpha\beta} + \frac{\partial f}{\partial B_{\alpha\beta}} D_{\alpha\beta} + \frac{\partial f}{\partial a_{\alpha}} c_{\alpha} + \frac{\partial f}{\partial b_{\alpha}} d_{\alpha} + \frac{\partial f}{\partial a} c. \tag{3.7}$$

* We note for future reference that if $\mathbf{A}$ is a symmetric second-order tensor, then $\partial f/\partial A_{\alpha\beta}$ stands for the symmetric form

$$\frac{1}{2}\left(\frac{\partial f}{\partial A_{\alpha\beta}} + \frac{\partial f}{\partial A_{\beta\alpha}}\right).$$

Finally, we will use the notation $\operatorname{div} \mathbf{q} = q^{\alpha}|_{\alpha}$ and grad T, with components $T_{,\alpha}$, for vector functions $\mathbf{q}$ and scalar functions T, respectively.

The *generalized stress* and *generalized strain* or, more simply, stress and strain, are functions whose values are elements of $\mathscr{a}$ and will be denoted by*

$$\begin{aligned} \mathbf{\Psi} &= (\mathbf{N}', \mathbf{M}, \mathbf{M}^3, \mathbf{m}, m^3), \\ \mathbf{\Phi} &= (\mathbf{e}, \boldsymbol{\kappa}, \boldsymbol{\kappa}_3, \boldsymbol{\delta}, \delta_3), \end{aligned} \tag{3.8}$$

respectively. Here $\mathbf{N}', \mathbf{M}, \mathbf{e}$ and $\boldsymbol{\kappa}$ are the second order tensors

$$\begin{aligned} \mathbf{N}' &= N'^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \qquad & \mathbf{M} &= M^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \\ \mathbf{e} &= e_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, & \boldsymbol{\kappa} &= \kappa_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \end{aligned} \tag{3.9}$$

where $\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$ denotes the tensor product of the vectors $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$; $\mathbf{M}^3, \mathbf{m}, \boldsymbol{\kappa}_3$ and $\boldsymbol{\delta}$ are the vectors

$$\mathbf{M}^3 = M^{3\alpha}\mathbf{a}_\alpha, \qquad \mathbf{m} = m^\alpha \mathbf{a}_\alpha, \qquad \boldsymbol{\kappa}_3 = \kappa_{3\alpha}\mathbf{a}^\alpha, \qquad \boldsymbol{\delta} = \delta_\alpha \mathbf{a}^\alpha, \tag{3.10}$$

and m^3 and δ_3 are scalars. The components of the various quantities in (3.9) and (3.10) are defined in equations (2.10), (2.12), (2.14), $(2.16)_{3,4}$ and $(2.17)_2$. Using the above notation, the energy equation (2.18) can be written as

$$-\rho(\dot{A} + T\dot{S} + S\dot{T}) + \mathbf{\Psi} \cdot \dot{\mathbf{\Phi}} + \rho r - \operatorname{div} \mathbf{q} = 0. \tag{3.11}$$

We now postulate the existence of a new kinematic variable $\mathbf{\Phi}''$, namely†

$$\mathbf{\Phi}'' = (\mathbf{e}'', \boldsymbol{\kappa}'', \boldsymbol{\kappa}_3'', \boldsymbol{\delta}'', \delta_3''). \tag{3.12}$$

We refer to $\mathbf{\Phi}''$ as the (generalized) *plastic strain* and assume that it possesses the same invariance properties under superposed rigid body motions as does $\mathbf{\Phi}$. Next, we introduce the following constitutive equation for the strain

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{\Psi}, \mathbf{\Phi}'', T) \tag{3.13}$$

and assume that for some range of values of the variables this equation has a unique inverse of the form‡

$$\mathbf{\Psi} = \mathbf{\Psi}(\mathbf{\Phi}, \mathbf{\Phi}'', T). \tag{3.14}$$

We now proceed to develop a constitutive equation for $\mathbf{\Phi}''$. For this purpose, we first introduce the concept of a loading function. We assume that at each instant of time there exists a scalar-valued continuously differentiable function f—called the loading function—of the instantaneous stress, plastic strain and temperature and write

$$f(\mathbf{\Psi}, \mathbf{\Phi}'', T) = \chi. \tag{3.15}$$

Equation (3.15) defines a hypersurface Σ (called the loading surface) in a thirteen dimensional Euclidean space $\mathscr{B}$ whose elements are the ordered pairs $(\mathbf{\Psi}, T)$. Of course, the surface Σ depends on $\mathbf{\Phi}''$ as a parameter. The scalar χ is a work-hardening parameter which is assumed to be initially positive and which depends on the past history of the

* Since $\mathbf{N}'$ and $\mathbf{e}$ are symmetric, $\mathbf{\Psi}$ and $\mathbf{\Phi}$ belong to a twelve dimensional subspace of $\mathscr{a}$.

† The tensor $\mathbf{e}$ is symmetric and we further suppose that $\mathbf{e}''$ is symmetric. For full generality $\mathbf{e}''$ should be a non-symmetric tensor, but this will not be considered here.

‡ We need only make the assumption (3.14) and omit (3.13) except in Section 6.

motion of the Cosserat surface. It is assumed that Σ is a closed surface and that the set of points

$$\sigma = \{(\boldsymbol{\Psi}, T) : f(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T) \leqq \chi\} \tag{3.16}$$

defines all possible stress–temperature pairs for the Cosserat surface $\mathscr{S}$ corresponding to the plastic strain $\boldsymbol{\Phi}''$. The boundary of σ is denoted by $\sigma_{\mathscr{B}}$ and is the set of points $(\boldsymbol{\Psi}, T) \in \mathscr{B}$ for which $f(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T) = \chi$. Similarly the interior of σ, σ_I, is the set of points $(\boldsymbol{\Psi}, T) \in \mathscr{B}$ for which $f(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T) < \chi$.

The surface Σ defined by (3.15) remains stationary (i.e., $\dot{\boldsymbol{\Phi}}'' = 0$ and $\dot{\chi} = 0$) as time progresses as long as $(\boldsymbol{\Psi}, T) \in \sigma_I$. For points in $\sigma_{\mathscr{B}}$, the surface remains stationary if

$$\left.\left(\frac{\partial f}{\partial \boldsymbol{\Psi}} \cdot \dot{\boldsymbol{\Psi}} + \frac{\partial f}{\partial T} \dot{T}\right)\right|_{(\boldsymbol{\Psi}, T) \in \sigma_{\mathscr{B}}} \leqq 0. \tag{3.17}$$

For convenience, we introduce the notation

$$\Delta f(\dot{\boldsymbol{\Psi}}, \dot{T}) = \frac{\partial f}{\partial \boldsymbol{\Psi}} \cdot \dot{\boldsymbol{\Psi}} + \frac{\partial f}{\partial T} \dot{T}, \tag{3.18}$$

and write the preceding statement as

$$\Delta f(\dot{\boldsymbol{\Psi}}, \dot{T})|_{(\boldsymbol{\Psi}, T) \in \sigma_{\mathscr{B}}} \leq 0 \Rightarrow \dot{\boldsymbol{\Phi}}'' = 0, \qquad \dot{\chi} = 0. \tag{3.19}$$

The strict inequality is referred to as unloading and the equality as neutral loading. On the other hand, corresponding to

$$\Delta f(\dot{\boldsymbol{\Psi}}, \dot{T})|_{(\boldsymbol{\Psi}, T) \in \sigma_{\mathscr{B}}} > 0 \tag{3.20}$$

we say loading occurs and we have

$$\Delta f(\dot{\boldsymbol{\Psi}}, \dot{T})|_{(\boldsymbol{\Psi}, T) \in \sigma_{\mathscr{B}}} > 0 \Rightarrow \dot{\boldsymbol{\Phi}}'' \neq 0. \tag{3.21}$$

We note that for the case of loading $\dot{\chi}$ may or may not vanish. During loading, we postulate the following constitutive equation for $\dot{\boldsymbol{\Phi}}''$:

$$\dot{\boldsymbol{\Phi}}'' = \boldsymbol{\Gamma}(\boldsymbol{\Psi}, \dot{\boldsymbol{\Psi}}, \boldsymbol{\Phi}'', T, \dot{T}). \tag{3.22}$$

Furthermore, we suppose that $\dot{\boldsymbol{\Phi}}''$ is linear in $\dot{\boldsymbol{\Psi}}$ and $\dot{T}$ so that

$$\dot{\boldsymbol{\Phi}}'' = \mathbf{L}(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T)[\dot{\boldsymbol{\Psi}}] + \boldsymbol{\Lambda}(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T)\dot{T}. \tag{3.23}$$

This is a fundamental assumption in the theory, as it means that the plastic strain depends on the ordering in time of the stress–temperature pairs, but is independent of the rate at which these pairs are traversed.

To complete our postulates concerning the loading function, we suppose that $\dot{\chi}$ is given by

$$\dot{\chi} = \zeta(\boldsymbol{\Psi}, \dot{\boldsymbol{\Psi}}, \boldsymbol{\Phi}'', \dot{\boldsymbol{\Phi}}'', T, \dot{T}) \tag{3.24}$$

and that ζ is linear in $\dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{\Phi}}''$, and $\dot{T}$. This condition again ensures that the constitutive equation is independent of the time scale used. Since $\dot{\chi}$ vanishes when $\dot{\boldsymbol{\Phi}}''$ vanishes, the constitutive equation for $\dot{\chi}$ reduces to

$$\dot{\chi} = \boldsymbol{\Delta}(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T) . \dot{\boldsymbol{\Phi}}''. \tag{3.25}$$

Assuming that the constitutive equation (3.23) is a continuous function of $(\mathbf{\Psi}, T)$, the transformation $\mathbf{L}$ and the function $\mathbf{\Lambda}$ must tend to zero as*

$$\Delta f(\dot{\mathbf{\Psi}}, \dot{T}) \to 0$$

which corresponds in the limit to neutral loading. Thus, we have

$$\mathbf{L}(\mathbf{\Psi}, \mathbf{\Phi}'', T)[\dot{\mathbf{\Psi}}] + \mathbf{\Lambda}(\mathbf{\Psi}, \mathbf{\Phi}'', T)\dot{T} = 0 \quad \text{whenever} \quad \Delta f(\dot{\mathbf{\Psi}}, \dot{T}) = 0. \tag{3.26}$$

Introducing the Lagrange multiplier† $\lambda\mathbf{B}$, where both λ and $\mathbf{B}$ are functions of $\mathbf{\Psi}$, $\mathbf{\Phi}''$ and T, and recalling the definition (3.18), we may combine the above two equations and write

$$\left(\mathbf{L} - \lambda\mathbf{B}\frac{\partial f}{\partial \mathbf{\Psi}}\right)[\dot{\mathbf{\Psi}}] + \left(\mathbf{\Lambda} - \lambda\mathbf{B}\frac{\partial f}{\partial T}\right)\dot{T} = 0. \tag{3.27}$$

Now, choose $\lambda\mathbf{B}$ so that the coefficient of $\dot{T}$ vanishes; then, since the remaining equation is valid for arbitrary stress rates (the coefficient being independent of rates), we have

$$\mathbf{L} = \lambda\mathbf{B}\frac{\partial f}{\partial \mathbf{\Psi}}, \qquad \mathbf{\Lambda} = \lambda\mathbf{B}\frac{\partial f}{\partial T}. \tag{3.28}$$

Again making use of definition (3.18) and using the above results, the expression for $\dot{\mathbf{\Phi}}''$ becomes

$$\dot{\mathbf{\Phi}}'' = \lambda\mathbf{B}\,\Delta f(\dot{\mathbf{\Psi}}, \dot{T}) \tag{3.29}$$

which holds during loading and neutral loading. During loading at least one component of $\dot{\mathbf{\Phi}}''$ must be non-zero, so that $\lambda \neq 0$. Therefore, without loss in generality, we may choose

$$\lambda > 0. \tag{3.30}$$

During loading,

$$\dot{f} = \frac{\partial f}{\partial \mathbf{\Psi}} \cdot \dot{\mathbf{\Psi}} + \frac{\partial f}{\partial \mathbf{\Phi}''} \cdot \dot{\mathbf{\Phi}}'' + \frac{\partial f}{\partial T}\dot{T} = \mathbf{\Lambda} \cdot \dot{\mathbf{\Phi}}'', \tag{3.31}$$

where we have used (3.25). Substituting (3.29) for $\dot{\mathbf{\Phi}}''$ and collecting terms, (3.31) yields

$$\left[\left(\mathbf{\Lambda} - \frac{\partial f}{\partial \mathbf{\Phi}''}\right) \cdot \lambda\mathbf{B} - 1\right] \Delta f(\dot{\mathbf{\Psi}}, \dot{T}) = 0, \tag{3.32}$$

which holds for all positive values of $\Delta f(\dot{\mathbf{\Psi}}, \dot{T})$. Since the quantity in brackets in (3.32) is independent of rates, we have

$$\left(\mathbf{\Lambda} - \frac{\partial f}{\partial \mathbf{\Phi}''}\right) \cdot \lambda\mathbf{B} = 1. \tag{3.33}$$

This equation may be regarded as defining λ.

By combining terms, (3.29) can now be written as

$$\dot{\mathbf{\Phi}}'' = \Lambda\mathbf{B}, \tag{3.34}$$

* For convenience, here and in later work we delete the subscript $(\mathbf{\Psi}, T) \in \sigma_{\mathscr{B}}$.

† Here λ is a scalar function and $\mathbf{B}$ is a function which has values in $\mathscr{a}$. We write the multiplier as a product only for convenience in later work.

where the scalar

$$\Lambda = \lambda\,\Delta f(\dot{\mathbf{\Psi}}, \dot{T}) \tag{3.35}$$

is positive during loading. Using (3.31), we may also write (3.35) in the alternative form

$$\Lambda = \lambda\left(\mathbf{\Delta} - \frac{\partial f}{\partial \mathbf{\Phi}''}\right) \cdot \dot{\mathbf{\Phi}}'', \tag{3.36}$$

or, using (3.34),

$$\Lambda = \frac{\dot{\mathbf{\Phi}}'' . \mathbf{B}}{\mathbf{B} . \mathbf{B}}. \tag{3.37}$$

4. THERMODYNAMIC RESTRICTIONS

We discuss in this Section the thermodynamic restrictions imposed on the constitutive equations of Section 3 by the Clausius–Duhem inequality. First, however, supplementary to the constitutive equations proposed in Section 3, we add the following constitutive postulates:

$$A = A(\mathbf{\Phi}, \mathbf{\Phi}'', T), \qquad S = S(\mathbf{\Phi}, \mathbf{\Phi}'', T), \qquad \mathbf{q} = \mathbf{q}(\mathbf{\Phi}, \mathbf{\Phi}'', T, \operatorname{grad} T). \tag{4.1}$$

These quantities also depend on the initial values of the kinematic variables at time t_0, but for convenience we will omit explicit reference to this dependence. We note that A, S and $\mathbf{q}$, in the form (4.1), remain invariant under all superposed rigid body motions.

Combination of the energy equation (3.11) and the inequality (2.19) yields

$$-\rho(\dot{A} + S\dot{T}) + \mathbf{\Psi} . \dot{\mathbf{\Phi}} - \frac{\mathbf{q} . \operatorname{grad} T}{T} \geq 0. \tag{4.2}$$

Substituting $(4.1)_1$ into (4.2), we obtain

$$-\rho\left(S + \frac{\partial A}{\partial T}\right)\dot{T} - \left(\rho\frac{\partial A}{\partial \mathbf{\Phi}} - \mathbf{\Psi}\right) \cdot \dot{\mathbf{\Phi}} - \rho\frac{\partial A}{\partial \mathbf{\Phi}''} \cdot \dot{\mathbf{\Phi}}'' - \frac{\mathbf{q} . \operatorname{grad} T}{T} \geq 0. \tag{4.3}$$

The inequality (4.3) holds for all arbitrary values of $\dot{T}$, $\dot{\mathbf{\Phi}}$ and for given values of T, $\mathbf{\Phi}$, $\mathbf{\Phi}''$ and is therefore valid during loading, unloading and neutral loading. In particular, during unloading ($\dot{\mathbf{\Phi}}'' = 0$), following the procedure of Coleman and Noll [5], we obtain

$$S = -\frac{\partial A}{\partial T}, \qquad \mathbf{\Psi} = \rho\frac{\partial A}{\partial \mathbf{\Phi}},$$
$$-\mathbf{q} . \operatorname{grad} T \geq 0. \tag{4.4}$$

Using (4.4), the energy equation reduces to

$$-\rho\frac{\partial A}{\partial \mathbf{\Phi}''} \cdot \dot{\mathbf{\Phi}}'' - \rho T\dot{S} + \rho r - \operatorname{div}\mathbf{q} = 0, \tag{4.5}$$

and (4.3) becomes

$$-\rho\frac{\partial A}{\partial \mathbf{\Phi}''} \cdot \dot{\mathbf{\Phi}}'' - \frac{\mathbf{q} . \operatorname{grad} T}{T} \geq 0. \tag{4.6}$$

It then follows that during both unloading and neutral loading, instead of (4.5) we have

$$-\rho T\dot{S}+\rho r-\operatorname{div}\mathbf{q} = 0. \tag{4.7}$$

During loading, with the help of (3.29), (4.6) yields

$$-\rho\lambda\frac{\partial A}{\partial\mathbf{\Phi}''}\cdot\mathbf{B}\,\Delta f(\dot{\mathbf{\Psi}},\dot{T})-\mathbf{q}\cdot\frac{\operatorname{grad}T}{T}\geq 0, \tag{4.8}$$

which must hold for arbitrary temperature fields and, in particular, for a homogeneous temperature distribution. Therefore, since we have taken $\lambda > 0$, we obtain the restriction

$$\frac{\partial A}{\partial\mathbf{\Phi}''}\cdot\mathbf{B}\leq 0. \tag{4.9}$$

We also specify here one additional property of the plastic strain, first introduced in (3.12). Supplementary to the various properties of $\mathbf{\Phi}''$ noted in Section 3, we require that upon return of T and $\mathbf{\Psi}$ to a state of uniform temperature T_0 and zero stress (not necessarily by unloading) in the neighborhood of a material point,

$$\mathbf{\Phi} = \mathbf{\Phi}'', \qquad S = S_0, \tag{4.10}$$

where S_0 is the entropy of the stress-free state at temperature T_0.

For later convenience, we close this Section with a list of constitutive equations for an elastic–plastic Cosserat surface. Thus,

$$\begin{aligned}
&f(\mathbf{\Psi},\mathbf{\Phi}'',T) = \chi, \qquad A = A(\mathbf{\Phi},\mathbf{\Phi}'',T),\\
&q^{\alpha} = q^{\alpha}(\mathbf{\Phi},\mathbf{\Phi}'',T,\operatorname{grad}T),\\
&\dot{\chi} = \overset{(1)}{H}{}^{\alpha\beta}\dot{e}''_{\alpha\beta}+\overset{(2)}{H}{}^{\alpha\beta}\dot{\kappa}''_{\alpha\beta}+\overset{(1)}{H}{}^{\alpha}\dot{\kappa}''_{3\alpha}+\overset{(2)}{H}{}^{\alpha}\dot{\delta}''_{\alpha}+H\dot{\delta}''_{3},\\
&\dot{e}''_{\alpha\beta} = \lambda\overset{(1)}{B}_{\alpha\beta}\,\Delta f(\dot{\mathbf{\Psi}},\dot{T}),\\
&\dot{\kappa}''_{\alpha\beta} = \lambda\overset{(2)}{B}_{\alpha\beta}\Delta f(\dot{\mathbf{\Psi}},\dot{T}), \qquad \dot{\kappa}''_{3\alpha} = \lambda\overset{(1)}{B}_{\alpha}\Delta f(\dot{\mathbf{\Psi}},\dot{T}),\\
&\dot{\delta}''_{\alpha} = \lambda\overset{(2)}{B}_{\alpha}\Delta f(\dot{\mathbf{\Psi}},\dot{T}), \qquad \dot{\delta}''_{3} = \lambda B\Delta f(\dot{\mathbf{\Psi}},\dot{T}),
\end{aligned} \tag{4.11}$$

In the above equations, $\overset{(\gamma)}{H}{}^{\alpha\beta}$, $\overset{(\gamma)}{B}_{\alpha\beta}$, $\overset{(\gamma)}{H}{}^{\alpha}$, $\overset{(\gamma)}{B}_{\alpha}$, $(\gamma = 1,2)$, H, B are all functions of $\mathbf{\Psi}$, $\mathbf{\Phi}''$, T and

$$\Delta f(\dot{\mathbf{\Psi}},\dot{T}) = \frac{\partial f}{\partial N'^{\alpha\beta}}\dot{N}'^{\alpha\beta}+\frac{\partial f}{\partial M^{\alpha\beta}}\dot{M}^{\alpha\beta}+\frac{\partial f}{\partial M^{3\beta}}\dot{M}^{3\beta}+\frac{\partial f}{\partial m^{\alpha}}\dot{m}^{\alpha}+\frac{\partial f}{\partial m^{3}}\dot{m}^{3}+\frac{\partial f}{\partial T}\dot{T}. \tag{4.12}$$

We also note that the component forms of equations (4.4) are

$$\begin{aligned}
S &= -\frac{\partial A}{\partial T}, \qquad N'^{\alpha\beta} = \rho\frac{\partial A}{\partial e_{\alpha\beta}}, \qquad M^{\alpha\beta} = \rho\frac{\partial A}{\partial\kappa_{\alpha\beta}},\\
M^{3\beta} &= \rho\frac{\partial A}{\partial\kappa_{3\beta}}, \qquad m^{\alpha} = \rho\frac{\partial A}{\partial\delta_{\alpha}}, \qquad m^{3} = \rho\frac{\partial A}{\partial\delta_{3}}.
\end{aligned} \tag{4.13}$$

5. AN ELASTIC-PERFECTLY PLASTIC COSSERAT SURFACE

A Cosserat surface will be called *elastic–perfectly plastic* if the loading function f and the function $\mathbf{B}$ reduce to the forms

$$f(\boldsymbol{\Psi}, T) = k, \tag{5.1}$$

and*

$$\mathbf{B} = \mathbf{B}(\boldsymbol{\Psi}, T), \tag{5.2}$$

where k is a real scalar. In the terminology of Section 3, since f is independent of $\boldsymbol{\Phi}''$, the hypersurface Σ is always stationary; we may, therefore, regard the elastic–perfectly plastic Cosserat surface as a limiting case of the theory developed in Sections 3 and 4. We note that all terms involving $\partial f/\partial\boldsymbol{\Phi}''$ must vanish and that $\Delta(\boldsymbol{\Psi}, \boldsymbol{\Phi}'', T)$, defined in (3.25), must vanish also. Moreover, in this case, neutral loading does not exist and during loading we must have

$$\Delta f(\dot{\boldsymbol{\Psi}}, \dot{T})|_{(\boldsymbol{\Psi}, T)\in\partial\mathscr{B}} = 0 \Rightarrow \dot{\boldsymbol{\Phi}}'' \neq 0. \tag{5.3}$$

It then follows from (3.33) that $\lambda \to \infty$ with $\mathbf{B}(\boldsymbol{\Psi}, T)$ remaining finite and, corresponding to (3.34), we write

$$\dot{\boldsymbol{\Phi}}'' = \tilde{\Lambda}\mathbf{B}(\boldsymbol{\Psi}, T), \tag{5.4}$$

where $\tilde{\Lambda}$ is given by

$$\tilde{\Lambda} = \frac{\dot{\boldsymbol{\Phi}}''.\mathbf{B}}{\mathbf{B}.\mathbf{B}}. \tag{5.5}$$

Equations (4.5) and (4.9) remain valid for the case of the elastic–perfectly plastic Cosserat surface, but (4.8) must be replaced by

$$-\rho\tilde{\Lambda}\frac{\partial A}{\partial\boldsymbol{\Phi}''}\cdot\mathbf{B} - \frac{\mathbf{q}\,.\,\text{grad}\,T}{T} \geq 0. \tag{5.6}$$

6. A RIGID-PLASTIC COSSERAT SURFACE

A useful idealization of elastic–plastic theory is the concept of rigid plasticity, i.e., a plasticity theory in which the total strain is identified with the plastic strain. Here we consider the case of a rigid–plastic Cosserat surface.

We define a rigid–plastic Cosserat surface to be one for which (3.13) reduces to

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}''. \tag{6.1}$$

In this case, the stress is no longer given by the constitutive assumption (3.14): however, the stress must still satisfy the equations of motion (2.16), the yield condition (3.15), and the constitutive equation (3.34) for the plastic strain rate.† The constitutive postulates

* Green and Naghdi, in Section 9 of [2], regard (5.1) alone as the defining condition for an elastic–perfectly plastic continuum. We make the further assumption that $\mathbf{B}$ is independent of $\boldsymbol{\Phi}''$, which may be regarded as a special case of Green and Naghdi's formulation.

† In certain special cases and with the neglect of inertia terms, the stress is statically determinate, i.e., it may be determined from the equilibrium equations and the yield condition alone.

(4.1) reduce to

$$\begin{aligned} A &= A(\mathbf{\Phi}'', T), \\ S &= S(\mathbf{\Phi}'', T), \\ \mathbf{q} &= \mathbf{q}(\mathbf{\Phi}'', T, \text{grad } T), \end{aligned} \tag{6.2}$$

and the inequality (4.3) becomes

$$-\rho\left(S+\frac{\partial A}{\partial T}\right)\dot{T}-\left(\rho\frac{\partial A}{\partial \mathbf{\Phi}''}-\mathbf{\Psi}\right)\cdot\dot{\mathbf{\Phi}}''-\frac{\mathbf{q}\cdot\text{grad } T}{T} \geq 0 \tag{6.3}$$

which must hold during loading, unloading, and neutral loading. During unloading ($\dot{\mathbf{\Phi}}'' = 0$), following the same procedure as before, we find

$$S = -\frac{\partial A}{\partial T}. \tag{6.4}$$

Then, the energy equation reduces to

$$-\left(\rho\frac{\partial A}{\partial \mathbf{\Phi}''}-\mathbf{\Psi}\right)\cdot\dot{\mathbf{\Phi}}''-\rho T\dot{S}+\rho r-\text{div}\,\mathbf{q} = 0, \tag{6.5}$$

and (4.3) becomes

$$-\left(\rho\frac{\partial A}{\partial \mathbf{\Phi}''}-\mathbf{\Psi}\right)\cdot\dot{\mathbf{\Phi}}''-\frac{\mathbf{q}\cdot\text{grad } T}{T} \geq 0. \tag{6.6}$$

In a manner similar to (4.9), from (3.34) and (6.6) we obtain the restriction

$$\left(\rho\frac{\partial A}{\partial \mathbf{\Phi}''}-\mathbf{\Psi}\right)\cdot\mathbf{B} \leq 0. \tag{6.7}$$

The above results for a work-hardening rigid–plastic Cosserat surface are obtained as a special case of the theory of Sections 3 and 4. The limiting case of a rigid–perfectly plastic surface can be discussed in a manner similar to that in Section 5, but we omit the details here.

7. A LINEAR THEORY OF AN ELASTIC–PERFECTLY PLASTIC COSSERAT SURFACE

In this Section we develop a linearized theory for an elastic–perfectly plastic Cosserat surface which has material symmetries corresponding to those of a three dimensional transversely isotropic shell. The procedure of linearization outlined here is similar to that in Section 6 of [1].

Let

$$\begin{aligned} \mathbf{r} &= \mathbf{R}+\varepsilon\mathbf{u}, & \mathbf{u} &= u^{\alpha}\mathbf{A}_{\alpha}+u^{3}\mathbf{A}_{3}, & \mathbf{v} &= \varepsilon\dot{\mathbf{u}}, \\ \mathbf{d} &= \mathbf{D}+\varepsilon\bar{\boldsymbol{\delta}}, & \bar{\boldsymbol{\delta}} &= \bar{\delta}^{\alpha}\mathbf{A}_{\alpha}+\bar{\delta}^{3}\mathbf{A}_{3}, & \mathbf{w} &= \varepsilon\dot{\bar{\boldsymbol{\delta}}}, \end{aligned} \tag{7.1}$$

where $\mathbf{R} = \mathbf{r}(x^{\alpha}, t_0)$, $\mathbf{D} = \mathbf{d}(x^{\alpha}, t_0)$, and ε is a non-dimensional parameter. We assume that the plastic strain and the plastic strain rate are of $O(\varepsilon)$, and that the temperature change

$(T-T_0)/T_0$ from a reference state is of $O(\varepsilon)$. Furthermore, we assume that the Cosserat surface is initially homogeneous, stress-free, and at rest with a uniform temperature T_0 and zero plastic strain. To obtain a linear theory, we neglect terms of $O(\varepsilon^3)$ in the energy equation and the loading surface, and after obtaining the desired approximations without loss in generality we set $\varepsilon = 1$. Thus, to the order of approximation considered, the kinematic variables (2.10) become*

$$
\begin{aligned}
2e_{\alpha\beta} &= u_{\alpha|\beta}+u_{\beta|\alpha}-2B_{\alpha\beta}u_3,\\
\kappa_{\alpha\beta} &= (\bar{\delta}_{\alpha|\beta}-B_{\alpha\beta}\bar{\delta}_3)+[u^{\nu}|_{\alpha}-B^{\nu}_{\alpha}u_3)(D_{\nu|\beta}-B_{\nu\beta}D_3)-\beta_{\alpha}(D_{3,\beta}+B^{\gamma}_{\beta}D_{\gamma})],\\
\kappa_{3\beta} &= (\bar{\delta}_{3,\beta}+B^{\gamma}_{\beta}\bar{\delta}_{\gamma})+\beta^{\gamma}(D_{\gamma|\beta}-B_{\gamma\beta}D_3),\\
\delta_{\alpha} &= \bar{\delta}_{\alpha}+D_{\gamma}(u^{\gamma}|_{\alpha}-B^{\gamma}_{\alpha}u_3)-\beta_{\alpha}D_3,\\
\delta_3 &= \bar{\delta}_3+\beta^{\alpha}D_{\alpha},
\end{aligned}
\tag{7.2}
$$

where

$$
\boldsymbol{\beta} = \beta_{\alpha}\mathbf{A}^{\alpha}, \qquad \beta_{\alpha} = -(u_{3,\alpha}+B^{\gamma}_{\alpha}u_{\gamma}),
\tag{7.3}
$$

and covariant differentiation is now with respect to $A_{\alpha\beta}$ of the undeformed initial surface.

We consider a special case of the infinitesimal theory when the initial director $\mathbf{D}$ is of constant length and coincident with the unit normal $\mathbf{A}_3$ to the initial surface $\mathscr{S}$ so that

$$
D_{\alpha} = 0, \qquad D_3 = 1.
\tag{7.4}
$$

Hence, from (2.9), we have

$$
\Lambda_{\alpha\beta} = -B_{\alpha\beta}, \qquad \Lambda_{3\beta} = 0,
\tag{7.5}
$$

which with (7.4) simplify some of the expressions in (7.2). Motivated by the existing classical theories of plates and shells, we introduce new kinematic variables $\rho_{\alpha\beta}$, $\rho_{3\beta}$ and stress variables V^{α}, V^3 by

$$
\begin{aligned}
\rho_{\alpha\beta} &= \kappa_{\alpha\beta}+B_{\alpha\beta}\delta_3, & \rho_{3\beta} &= \kappa_{3\beta}-B^{\gamma}_{\beta}\delta_{\gamma},\\
V^{\alpha} &= m^{\alpha}+B^{\alpha}_{\beta}M^{3\beta}, & V^3 &= m^3-B_{\alpha\beta}M^{\beta\alpha}.
\end{aligned}
\tag{7.6}
$$

Also, using the special values (7.4), the kinematic variables can be written as

$$
\begin{aligned}
\delta_{\alpha} &= \bar{\delta}_{\alpha}-\beta_{\alpha}, \qquad \delta_3 = \bar{\delta}_3,\\
\rho_{\alpha\beta} &= \bar{\delta}_{\alpha|\beta}-B_{\beta\gamma}u^{\gamma}|_{\alpha}+B_{\beta\gamma}B^{\gamma}_{\alpha}u_3\\
&= \delta_{\alpha|\beta}-[u^3|_{\alpha\beta}+B_{\alpha\gamma|\beta}u^{\gamma}+B_{\alpha\gamma}u^{\gamma}|_{\beta}+B_{\beta\gamma}u^{\gamma}|_{\alpha}-B_{\beta\gamma}B^{\gamma}_{\alpha}u_3],\\
\rho_{3\beta} &= \bar{\delta}_{3,\beta} = \delta_{3,\beta}
\end{aligned}
\tag{7.7}
$$

with $e_{\alpha\beta}$ in $(7.2)_1$ remaining unchanged.

Using $(7.6)_{3,4}$ and (2.16), the equations of motion of the infinitesimal theory are

$$
\begin{aligned}
N^{\alpha\beta}|_{\beta}-B^{\alpha}_{\beta}N^{3\beta}+\rho_0F^{\alpha} &= \rho_0c^{\alpha}, & N^{3\beta}|_{\beta}+B_{\alpha\beta}N^{\beta\alpha}+\rho_0F^3 &= \rho_0c^3,\\
M^{\alpha\beta}|_{\beta}+\rho_0\bar{L}^{\alpha} &= V^{\alpha}, & M^{3\beta}|_{\beta}+\rho_0\bar{L}^3 &= V^3,
\end{aligned}
\tag{7.8}
$$

* For details, see Section 6 of [1].

where covariant differentiation is with respect to $A_{\alpha\beta}$ of the undeformed initial surface and ρ_0 is the mass density of the initial surface. Also, equations (2.17) reduce to

$$N^{3\beta} = V^\beta, \qquad N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} + M^{\alpha\gamma}B^\beta_\gamma, \tag{7.9}$$

and the energy equation (2.18) becomes

$$-\rho_0(\dot{A} + T\dot{S} + S\dot{T}) + N'^{\alpha\beta}\dot{e}_{\alpha\beta} + M^{\alpha\beta}\dot{\rho}_{\alpha\beta} + M^{3\beta}\dot{\rho}_{3\beta} + V^\alpha\dot{\delta}_\alpha + V^3\dot{\delta}_3 + \rho r - q^\alpha|_\alpha = 0. \tag{7.10}$$

Finally, with the help of (4.13) and the new variables (7.6), the constitutive equations of the linear theory are

$$\begin{gathered} S = -\frac{\partial A}{\partial T}, \qquad N'^{\alpha\beta} = \rho_0 \frac{\partial A}{\partial e_{\alpha\beta}}, \qquad M^{\alpha\beta} = \rho_0 \frac{\partial A}{\partial \rho_{\alpha\beta}}, \\ M^{3\beta} = \rho_0 \frac{\partial A}{\partial \rho_{3\beta}}, \qquad V^\alpha = \rho_0 \frac{\partial A}{\partial \delta_\alpha}, \qquad V^3 = \rho_0 \frac{\partial A}{\partial \delta_3}. \end{gathered} \tag{7.11}$$

For convenience, in what follows, we will collectively denote the stresses and strains in the notation of Section 3 by $\mathbf{\Phi}$ and $\mathbf{\Psi}$ which now stand for

$$\begin{aligned} \mathbf{\Phi} &= (\mathbf{e}, \boldsymbol{\rho}, \boldsymbol{\rho}_3, \boldsymbol{\delta}, \delta_3), \\ \mathbf{\Psi} &= (\mathbf{N}', \mathbf{M}, \mathbf{M}^3, \mathbf{V}, V^3). \end{aligned} \tag{7.12}$$

The Helmholtz free energy is a function of the temperature T, the strain $\mathbf{\Phi}$, the plastic strain $\mathbf{\Phi}''$, and the initial values $A_{\alpha\beta}$, $\Lambda_{\alpha\beta}$, $\Lambda_{3\beta}$, D_α and D_3. However, based on the special values (7.4) and the result (7.5), this list reduces to T, $\mathbf{\Phi}$, $\mathbf{\Phi}''$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$. When the free energy is taken to be a function of these variables, many of the classical linear theories of elastic shells (derived from the three dimensional equations of classical elasticity) can be shown to be equivalent to the linear theory of an elastic Cosserat surface,* also contained in the theory of this section with $\mathbf{\Phi}'' = 0$.

We now restrict our attention to a free energy function which is independent of $B_{\alpha\beta}$. Then, for a Cosserat surface which is isotropic with a center of symmetry, the free energy can be expressed in the form

$$\rho_0 A = \rho_0 A'(\mathbf{\Phi}, T) + \rho_0 A''(\mathbf{\Phi}'', T) + \rho_0 A'''(\mathbf{\Phi}, \mathbf{\Phi}''). \tag{7.13}$$

Here A' is a function of the joint invariants of $\mathbf{\Phi}$ and T, A'' is a function of the joint invariants of $\mathbf{\Phi}''$ and T, and A''' is a function of the joint invariants of $\mathbf{\Phi}$ and $\mathbf{\Phi}''$. In order to imitate the symmetries associated with a three dimensional shell which is transversely isotropic with respect to normals to the shell middle surface, we stipulate the additional requirement that the free energy must remain invariant under the transformations

$$\bar{\delta}_\alpha \to -\bar{\delta}_\alpha, \qquad \bar{\delta}_3 \to \bar{\delta}_3, \qquad u_\alpha \to u_\alpha, \qquad u_3 \to -u_3, \qquad B_{\alpha\beta} \to -B_{\alpha\beta}. \tag{7.14}$$

It then follows from (7.7) that

$$\rho_{\alpha\beta} \to -\rho_{\alpha\beta}, \qquad \delta_\alpha \to -\delta_\alpha, \tag{7.15}$$

with $e_{\alpha\beta}$, $\rho_{3\beta}$ and δ_3 remaining unchanged. We further assume that the transformations (7.14) imply that

$$\rho''_{\alpha\beta} \to -\rho''_{\alpha\beta}, \qquad \delta''_\alpha \to -\delta''_\alpha, \tag{7.16}$$

* See Green and Naghdi [6] for a discussion of particular cases.

with $e''_{\alpha\beta}$, $\rho''_{3\beta}$ and δ''_3 remaining unchanged. Since we have assumed that the Cosserat surface is initially homogeneous, stress-free, and at rest with a uniform temperature T_0 and zero plastic strain, it is sufficient to express the three functions on the right-hand side of (7.13) in the forms*

$$\begin{aligned}\rho_0 A' &= \tfrac{1}{2}[\alpha_1 A^{\alpha\beta}A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e_{\gamma\delta}\\ &\quad+\tfrac{1}{2}\alpha_3 A^{\alpha\beta}\delta_\alpha\delta_\beta+\tfrac{1}{2}\alpha_4(\delta_3)^2+\alpha'_4\delta_3 T+\tfrac{1}{2}\alpha''_4 T^2\\ &\quad+\tfrac{1}{2}[\alpha_5 A^{\alpha\beta}A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma}A^{\beta\delta}+\alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\alpha\beta}\rho_{\gamma\delta}\\ &\quad+\tfrac{1}{2}\alpha_8 A^{\alpha\beta}\rho_{3\alpha}\rho_{3\beta}+\alpha_9 A^{\alpha\beta}e_{\alpha\beta}\delta_3+\alpha'_9 A^{\alpha\beta}e_{\alpha\beta}T,\end{aligned} \tag{7.17a}$$

$$\begin{aligned}\rho_0 A'' &= \tfrac{1}{2}[\beta_1 A^{\alpha\beta}A^{\gamma\delta}+\beta_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e''_{\alpha\beta}e''_{\gamma\delta}\\ &\quad+\tfrac{1}{2}\beta_3 A^{\alpha\beta}\delta''_\alpha\delta''_\beta+\tfrac{1}{2}\beta_4(\delta''_3)^2+\beta'_4\delta''_3 T\\ &\quad+\tfrac{1}{2}[\beta_5 A^{\alpha\beta}A^{\gamma\delta}+\beta_6 A^{\alpha\gamma}A^{\beta\delta}+\beta_7 A^{\alpha\delta}A^{\beta\gamma}]\rho''_{\alpha\beta}\rho''_{\gamma\delta}\\ &\quad+\tfrac{1}{2}\beta_8 A^{\alpha\beta}\rho''_{3\alpha}\rho''_{3\beta}+\beta_9 A^{\alpha\beta}e''_{\alpha\beta}\delta''_3+\beta'_9 A^{\alpha\beta}e''_{\alpha\beta}T,\end{aligned} \tag{7.17b}$$

$$\begin{aligned}\rho_0 A''' &= [\gamma_1 A^{\alpha\beta}A^{\gamma\delta}+\gamma_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e''_{\gamma\delta}\\ &\quad+\gamma_3 A^{\alpha\beta}\delta_\alpha\delta''_\beta+\gamma_4\delta_3\delta''_3\\ &\quad+[\gamma_5 A^{\alpha\beta}A^{\gamma\delta}+\gamma_6 A^{\alpha\gamma}A^{\beta\delta}+\gamma_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\alpha\beta}\rho''_{\gamma\delta}\\ &\quad+\gamma_8 A^{\alpha\beta}\rho_{3\alpha}\rho''_{3\beta}+\gamma_9 A^{\alpha\beta}e_{\alpha\beta}\delta''_3+\gamma'_9 A^{\alpha\beta}e''_{\alpha\beta}\delta_3.\end{aligned} \tag{7.17c}$$

The stresses and entropy calculated from (7.11) and (7.17) will be linear combinations of the strains, plastic strains, and temperature. Recalling that we have already taken $S_0 = 0$, if we make use of the requirement [stated above (4.10)] that in the neighborhood of any point $\mathbf{\Phi} = \mathbf{\Phi}''$ and $S = 0$ when all stresses vanish and when $T = T_0$, then it follows that

$$\begin{aligned}\alpha_i &= -\gamma_i\,(i = 1, 2, \ldots, 9),\\ \alpha'_4 &= -\beta'_4, \qquad \alpha'_9 = -\beta'_9, \qquad \alpha_9 = -\gamma'_9.\end{aligned} \tag{7.18}$$

It is convenient to introduce

$$\mathbf{\Phi}' = \mathbf{\Phi} - \mathbf{\Phi}'', \tag{7.19}$$

which may be regarded as corresponding to the infinitesimal elastic strain† in the classical linear theory of elastic–plastic continua. With the notation (7.19), the expressions for the entropy and the stresses are

$$\begin{aligned}-\rho_0 S &= \alpha'_4\delta'_3+\alpha''_4 T+\alpha'_9 A^{\alpha\beta}e'_{\alpha\beta},\\ N'^{\alpha\beta} &= [\alpha_1 A^{\alpha\beta}A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e'_{\gamma\delta}+\alpha_9 A^{\alpha\beta}\delta'_3+\alpha'_9 A^{\alpha\beta}T,\\ M^{\alpha\beta} &= [\alpha_5 A^{\alpha\beta}A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma}A^{\beta\delta}+\alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho'_{\gamma\delta},\\ M^{3\beta} &= \alpha_8 A^{\beta\gamma}\rho'_{3\gamma}, \qquad V^\alpha = \alpha_3 A^{\alpha\beta}\delta'_\beta, \qquad V^3 = \alpha_4\delta'_3+\alpha'_4 T+\alpha_9 A^{\alpha\beta}e'_{\alpha\beta}.\end{aligned} \tag{7.20}$$

* In (7.17), T designates the temperature difference from T_0 and, for convenience, we have assumed that the initial entropy S_0 is zero. The expansion procedure used here is similar to that used in Section 6 of [1] with the exception that invariance under the transformations (7.15) and (7.16) is also assumed.

† Although the definition (7.19) can also be introduced in the development of the nonlinear theory of elastic–plastic continua, an unambiguous usage of the term "elastic strain" is applicable only to the infinitesimal theory. In either case, we only need use the total strain and the plastic strain in the development of the theory, the introduction of (7.19) being purely a matter of convenience.

We also note that the free energy, in view of (7.18), can be written in the form

$$\rho_0 A = \rho_0 A_1(\mathbf{\Phi}', T) + \rho_0 A_2(\mathbf{\Phi}''), \tag{7.21}$$

where $A_1(\mathbf{\Phi}', T)$ is a homogeneous quadratic form in $\mathbf{\Phi}'$ and T with the same coefficients $\alpha_1, \alpha_2, \ldots, \alpha_7, \alpha_4', \alpha_4'', \alpha_9'$ as in $A'(\mathbf{\Phi}, T)$ in (7.17a).

The expression for the loading function can be developed in a similar fashion to that of the Helmholtz free energy. Since the transformations (7.15) and (7.16) imply that $M^{\alpha\beta} \to -M^{\alpha\beta}$ and $V^\alpha \to -V^\alpha$, with $N'^{\alpha\beta}$, $M^{3\beta}$ and V^3 remaining unchanged, we also assume that the loading function must be invariant under these transformations. Thus, for a sufficiently smooth loading function, we have

$$\begin{aligned} f = {} & \tfrac{1}{2}[\zeta_1 A_{\alpha\beta}A_{\gamma\delta} + \zeta_2(A_{\alpha\gamma}A_{\beta\delta} + A_{\alpha\delta}A_{\beta\gamma})]N'^{\alpha\beta}N'^{\gamma\delta} \\ & + \tfrac{1}{2}\zeta_3 A_{\alpha\beta}V^\alpha V^\beta + \tfrac{1}{2}\zeta_4(V^3)^2 + \zeta_4' V^3 T + \tfrac{1}{2}\zeta_4'' T^2 \\ & + \tfrac{1}{2}[\zeta_5 A_{\alpha\beta}A_{\gamma\delta} + \zeta_6 A_{\alpha\gamma}A_{\beta\delta} + \zeta_7 A_{\alpha\delta}A_{\beta\gamma}]M^{\alpha\beta}M^{\gamma\delta} \\ & + \tfrac{1}{2}\zeta_8 A_{\alpha\beta}M^{3\alpha}M^{3\beta} + \zeta_9 A_{\alpha\beta}N'^{\alpha\beta}V^3 + \zeta_9' A_{\alpha\beta}N'^{\alpha\beta}T, \end{aligned} \tag{7.22}$$

where the coefficients $\zeta_1 \ldots \zeta_9'$ are constants.

The constitutive equations for the plastic strain rates are obtained by expanding the function $\mathbf{B}$ in (5.2) and substituting the resulting expression in (5.4). Motivated by our previous assumptions concerning invariance under the transformations (7.14)–(7.16) we see that $\dot{\rho}''_{\alpha\beta} \to -\dot{\rho}''_{\alpha\beta}$ and $\dot{\delta}''_\alpha \to -\dot{\delta}''_\alpha$ when $M^{\alpha\beta} \to -M^{\alpha\beta}$ and $V^\alpha \to -V^\alpha$, with all other strain rates remaining unchanged. With these conditions, the constitutive equations for $\overset{(\gamma)}{B}_{\alpha\beta}$, $\overset{(\gamma)}{B}_\alpha$ and B are

$$\begin{aligned} \overset{(1)}{B}_{\alpha\beta} &= [\eta_1 A_{\alpha\beta}A_{\gamma\delta} + \eta_2(A_{\alpha\gamma}A_{\beta\delta} + A_{\alpha\delta}A_{\beta\gamma})]N'^{\gamma\delta} + \eta_9 A_{\alpha\beta}V^3 + \eta_9' A_{\alpha\beta}T, \\ \overset{(2)}{B}_{\alpha\beta} &= [\eta_5 A_{\alpha\beta}A_{\gamma\delta} + \eta_6 A_{\alpha\gamma}A_{\beta\delta} + \eta_7 A_{\alpha\delta}A_{\beta\gamma}]M^{\gamma\delta}, \\ \overset{(1)}{B}_\alpha &= \eta_8 A_{\alpha\beta}M^{3\beta}, \qquad \overset{(2)}{B}_\alpha = \eta_3 A_{\alpha\beta}V^\beta, \qquad B = \eta_{10}A_{\alpha\beta}N'^{\alpha\beta} + \eta_4 V^3 + \eta_4' T, \end{aligned} \tag{7.23}$$

where the coefficients $\eta_1 \ldots \eta_4'$ are constants. With the help of (7.21), the inequality (4.9) becomes

$$-\mathbf{\Psi}\cdot\mathbf{B} + \frac{\partial A_2}{\partial \mathbf{\Phi}''}\cdot\mathbf{B} \le 0, \tag{7.24}$$

from which we also deduce the restrictions

$$\begin{aligned} & \eta_1+\eta_2 \ge 0, \quad \eta_2 \ge 0, \quad \eta_3 \ge 0, \quad 2\eta_5+\eta_6+\eta_7 \ge 0, \quad \eta_6 \ge 0, \\ & \eta_6+\eta_7 \ge 0, \quad \eta_6-\eta_7 \ge 0, \quad \eta_8 \ge 0, \quad \eta_4(\eta_1+\eta_2) \ge \tfrac{1}{4}(\eta_9+\eta_{10})^2, \\ & \eta_4' = 0, \quad \eta_9' = 0, \quad A_2(\mathbf{\Phi}'') \equiv 0. \end{aligned} \tag{7.25}$$

Note that the last result is a consequence of our assumption (5.2) that $\mathbf{B}$ is independent of $\mathbf{\Phi}''$.

Combining (7.23) and (7.25) with (5.4), the constitutive equations for the plastic strain rates may be expressed in the forms

$$\begin{aligned}
\dot{e}''_{\alpha\beta} &= \tilde{\Lambda}\{[\eta_1 A_{\alpha\beta}A_{\gamma\delta}+\eta_2(A_{\alpha\gamma}A_{\beta\delta}+A_{\alpha\delta}A_{\beta\gamma})]N'^{\gamma\delta}+\eta_9 A_{\alpha\beta}V^3\},\\
\dot{\rho}''_{\alpha\beta} &= \tilde{\Lambda}[\eta_5 A_{\alpha\beta}A_{\gamma\delta}+\eta_6 A_{\alpha\gamma}A_{\beta\delta}+\eta_7 A_{\alpha\delta}A_{\beta\gamma}]M^{\gamma\delta},\\
\dot{\rho}''_{3\beta} &= \tilde{\Lambda}\eta_8 A_{\beta\gamma}M^{3\gamma}, \qquad \dot{\delta}''_\alpha = \tilde{\Lambda}\eta_3 A_{\alpha\beta}V^\beta, \qquad \dot{\delta}''_3 = \tilde{\Lambda}[\eta_{10}A_{\alpha\beta}N'^{\alpha\beta}+\eta_4 V^3].
\end{aligned} \tag{7.26}$$

Finally, the equation for the heat flux vector is given by

$$q^\alpha = A^{\alpha\beta}[(\sigma_1\rho_{3\beta}+\xi_1\rho''_{3\beta})+(\sigma_2\delta_\beta+\xi_2\delta''_\beta)+\sigma_3 T_{,\beta}] \tag{7.27}$$

which must hold during loading and unloading. In particular, it must satisfy $(4.4)_3$ so that

$$\begin{gathered}
\sigma_1 = \sigma_2 = \xi_1 = \xi_2 = 0,\\
\sigma_3 \le 0,
\end{gathered} \tag{7.28}$$

and (7.27) reduces to Fourier's law, namely

$$q^\alpha = \sigma_3 A^{\alpha\beta}T_{,\beta}. \tag{7.29}$$

8. A LINEAR THEORY OF AN ELASTIC–PERFECTLY PLASTIC COSSERAT PLATE

A linear theory of an elastic Cosserat plate which in three dimensions is transversely isotropic has been developed by Green and Naghdi [7] as a special case of the general theory of a Cosserat surface. Here we treat the elastic–perfectly plastic Cosserat plate as a special case of the theory developed in Section 7.

It is convenient to refer the initially flat surface to a fixed system of rectangular Cartesian coordinates and designate the Cartesian components of **r** and **R** by z_i and Z_i $(i = 1, 2, 3)$, respectively. We quote from the infinitesimal theory of Section 7, after specializing the results to an initially flat surface and referring all quantities to rectangular Cartesian coordinates. Thus, from $(7.2)_1$ and (7.7), we obtain

$$\begin{gathered}
2e_{\alpha\beta} = u_{\alpha,\beta}+u_{\beta,\alpha},\\
\rho_{\alpha\beta} = \bar{\delta}_{\alpha,\beta} = \delta_{\alpha,\beta}-u_{3,\alpha\beta}, \qquad \rho_{3\beta} = \bar{\delta}_{3,\beta} = \delta_{3,\beta},\\
\delta_\alpha = \bar{\delta}_\alpha+u_{3,\alpha}, \qquad \delta_3 = \bar{\delta}_3,
\end{gathered} \tag{8.1}$$

where in (8.1) and throughout this Section a comma denotes partial differentiation with respect to Z_α. Also $(7.9)_2$ reduces to

$$N'_{\alpha\beta} = N'_{\beta\alpha} = N_{\alpha\beta}. \tag{8.2}$$

For the particular theory developed here, the full set of field equations separate into those for extensional deformation and those for bending deformation. We summarize these equations below:

Extensional theory

$$
\begin{aligned}
2e_{\alpha\beta} &= u_{\alpha,\beta}+u_{\beta,\alpha}, \qquad \rho_{3\beta} = \bar{\delta}_{3,\beta}, \qquad \delta_3 = \bar{\delta}_3,\\
e_{\alpha\beta} &= e'_{\alpha\beta}+e''_{\alpha\beta}, \qquad \rho_{3\beta} = \rho'_{3\beta}+\rho''_{3\beta}, \qquad \delta_3 = \delta'_3+\delta''_3,\\
\dot{e}''_{\alpha\beta} &= \tilde{\Lambda}[\eta_1\delta_{\alpha\beta}N_{\gamma\gamma}+2\eta_2N_{\alpha\beta}+\eta_9\delta_{\alpha\beta}V_3],\\
\dot{\rho}''_{3\beta} &= \tilde{\Lambda}\eta_8M_{3\beta}, \qquad \dot{\delta}''_3 = \tilde{\Lambda}[\eta_{10}N_{\gamma\gamma}+\eta_4V_3],\\
N_{\alpha\beta} &= \alpha_1\delta_{\alpha\beta}e'_{\gamma\gamma}+2\alpha_2e'_{\alpha\beta}+\alpha_9\delta_{\alpha\beta}\delta'_3+\alpha'_9\delta_{\alpha\beta}T,\\
M_{3\beta} &= \alpha_8\rho'_{3\beta}, \qquad V_3 = \alpha_4\delta'_3+\alpha_9e'_{\gamma\gamma}+\alpha'_4T,\\
q_\alpha &= \sigma_3T_{,\alpha}, \qquad -\rho_0S = \alpha'_4\delta'_3+\alpha''_4T+\alpha'_9e'_{\gamma\gamma},\\
&-\rho_0T_0\dot{S}+\rho_0r-q_{\alpha,\alpha} = 0,\\
&N_{\alpha\beta,\beta}+\rho_0F_\alpha = \rho_0c_\alpha, \qquad M_{3\beta,\beta}+\rho_0\bar{L}_3 = V_3,\\
f_e &= \tfrac{1}{2}\zeta_1N_{\alpha\alpha}N_{\beta\beta}+\zeta_2N_{\alpha\beta}N_{\alpha\beta}+\tfrac{1}{2}\zeta_4(V_3)^2+\zeta'_4V_3T\\
&\quad+\tfrac{1}{2}\zeta''_4T^2+\tfrac{1}{2}\zeta_8M_{3\alpha}M_{3\alpha}+\zeta_9N_{\alpha\alpha}V_3+\zeta'_9N_{\alpha\alpha}T = k.
\end{aligned}
\tag{8.3}
$$

The notation f_e is used to denote the extensional part of the loading function in (7.22).

Bending theory

$$
\begin{aligned}
\rho_{\alpha\beta} &= \rho_{(\alpha\beta)}+\rho_{[\alpha\beta]},\\
\rho_{(\alpha\beta)} &= \tfrac{1}{2}(\bar{\delta}_{\alpha,\beta}+\bar{\delta}_{\beta,\alpha}) = \tfrac{1}{2}(\delta_{\alpha,\beta}+\delta_{\beta,\alpha})-u_{3,\alpha\beta},\\
\rho_{[\alpha\beta]} &= \tfrac{1}{2}(\bar{\delta}_{\alpha,\beta}-\bar{\delta}_{\beta,\alpha}) = \tfrac{1}{2}(\delta_{\alpha,\beta}-\delta_{\beta,\alpha}), \qquad \delta_\alpha = \bar{\delta}_\alpha+u_{3,\alpha},\\
\rho_{(\alpha\beta)} &= \rho'_{(\alpha\beta)}+\rho''_{(\alpha\beta)}, \qquad \rho_{[\alpha\beta]} = \rho'_{[\alpha\beta]}+\rho''_{[\alpha\beta]}, \qquad \delta_\alpha = \delta'_\alpha+\delta''_\alpha,\\
\dot{\rho}''_{(\alpha\beta)} &= \tilde{\Lambda}[\eta_5\delta_{\alpha\beta}M_{\gamma\gamma}+(\eta_6+\eta_7)M_{(\alpha\beta)}],\\
\dot{\rho}''_{[\alpha\beta]} &= \tilde{\Lambda}(\eta_6-\eta_7)M_{[\alpha\beta]}, \qquad \dot{\delta}''_\alpha = \tilde{\Lambda}\eta_3V_\alpha,\\
M_{(\alpha\beta)} &= \alpha_5\delta_{\alpha\beta}\rho'_{\gamma\gamma}+(\alpha_6+\alpha_7)\rho'_{(\alpha\beta)},\\
M_{[\alpha\beta]} &= (\alpha_6-\alpha_7)\rho'_{[\alpha\beta]}, \qquad V_\alpha = \alpha_3\delta'_\alpha,\\
M_{\alpha\beta,\beta} &+\rho_0\bar{L}_\alpha = V_\alpha, \qquad V_{\alpha,\alpha}+\rho_0F_3 = \rho_0c_3,\\
f_b &= \tfrac{1}{2}\zeta_3V_\alpha V_\alpha+\tfrac{1}{2}\zeta_5M_{\alpha\alpha}M_{\beta\beta}+\tfrac{1}{2}(\zeta_6+\zeta_7)M_{(\alpha\beta)}M_{(\alpha\beta)}+\tfrac{1}{2}(\zeta_6-\zeta_7)M_{[\alpha\beta]}M_{[\alpha\beta]} = k.
\end{aligned}
\tag{8.4}
$$

Here f_b denotes the pure bending part of the loading function in (7.22).

9. IDENTIFICATION OF COEFFICIENTS AND SPECIAL CASES

In this section we consider the isothermal linear theory of an elastic–perfectly plastic Cosserat plate. In particular, using the results of Section 8, we relate some of the constitutive coefficients to corresponding known coefficients in the three dimensional theory.

Consider a three dimensional rectangular plate referred to a fixed rectangular Cartesian coordinate system Z_i ($i = 1, 2, 3$). Let the middle plane of the plate be defined by $Z_3 = 0$, designate the uniform thickness of the plate by h, and denote the Cartesian components of the symmetric stress tensor of the classical three dimensional continuum (with infinitesimal deformation) by $\sigma_{ij}\,(i, j = 1, 2, 3)$. By identifying the stress resultant

$$\int_{-h/2}^{h/2} \sigma_{\alpha\beta} Z_3 \, \mathrm{d}Z_3 \tag{9.1}$$

with $M_{(\alpha\beta)}$ and comparing solutions for the problem of simple flexure of a plate (which in three dimensions is isotropic), Green and Naghdi [7] have made the identification

$$\alpha_5 = \nu B, \qquad \alpha_6 + \alpha_7 = (1-\nu)B, \tag{9.2}$$

where $B = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate, E is Young's modulus and ν Poisson's ratio. Moreover, since in the classical plate theory (which is derived from three dimensional equations) $M_{\alpha\beta}$ is symmetric, we may also assume that *

$$\alpha_6 = \alpha_7. \tag{9.3}$$

By making the further identification of

$$\int_{-h/2}^{h/2} \sigma_{\alpha\beta} \, \mathrm{d}Z_3, \qquad \int_{-h/2}^{h/2} \sigma_{3\beta} Z_3 \, \mathrm{d}Z_3, \qquad \int_{-h/2}^{h/2} \sigma_{33} \, \mathrm{d}Z_3 \tag{9.4}$$

with $N_{\alpha\beta}$, $M_{3\beta}$ and V_3 respectively, and considering the case of simple extension it is possible to identify the following additional coefficients†:

$$\alpha_1 = \alpha_9 = \frac{(1-\nu)\nu D}{1-2\nu}, \qquad \alpha_2 = \frac{(1-\nu)D}{2}, \qquad \alpha_4 = \frac{(1-\nu)^2 D}{1-2\nu}, \tag{9.5}$$

where $D = Eh/(1-\nu^2)$. The two other coefficients in the stress–strain relations, namely α_3 and α_8, remain arbitrary. In the remainder of this Section we adopt the above identifications. Also, from (7.22) and (7.23), we have

$$\zeta_6 = \zeta_7, \qquad \eta_6 = \eta_7, \tag{9.6}$$

in view of (9.3).

It is possible to identify most of the coefficients which appear in (7.22) by comparison with a yield function of the isothermal linear three dimensional theory of an elastic–perfectly plastic continuum. We assume that the yield function of the three dimensional theory is sufficiently smooth and is also independent of the mean stress σ_{kk}. Put

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \qquad s_{ii} = 0. \tag{9.7}$$

Then, for an isotropic material with a center of symmetry, we have the von Mises yield condition given by

$$s_{ij}s_{ij} = k'. \tag{9.8}$$

* We note, however, that there may well be situations for which the identification (9.3) is not desirable.
† See Green and Naghdi [6].

Now by using the approximations

$$\sigma_{\alpha\beta} = \frac{1}{h}N_{\alpha\beta}+\frac{12}{h^3}M_{\alpha\beta}Z_3,$$
$$\sigma_{3\beta} = \frac{1}{h}V_\beta+\frac{12}{h^3}M_{3\beta}Z_3, \tag{9.9}$$
$$\sigma_{33} = \frac{1}{h}V_3$$

in (9.8) and integrating the resulting equation over the thickness of the plate, we can make the identifications:

$$k = k'h, \quad \zeta_1 = -\frac{2}{3h}, \quad \zeta_2 = \frac{1}{h}, \quad \zeta_3 = \frac{4}{h}, \quad \zeta_4 = \frac{4}{3h},$$
$$\zeta_5 = -\frac{8}{h^3}, \quad \zeta_6 = \zeta_7 = \frac{12}{h^3}, \quad \zeta_8 = \frac{48}{h^3}, \quad \zeta_9 = -\frac{2}{3h}. \tag{9.10}$$

Finally we introduce the special constitutive assumption*

$$\mathbf{B} = \frac{\partial f}{\partial \mathbf{\Psi}}. \tag{9.11}$$

It follows from (9.11), (7.22) and (7.26) that for the isothermal theory

$$\eta_i = \zeta_i \, (i = 1, 2, \ldots, 9),$$
$$\eta_{10} = \eta_9 = \zeta_9. \tag{9.12}$$

For the particular case of the von Mises yield condition these coefficients are given by (9.10). We note that the thermodynamic restrictions (7.25) are fulfilled.

The bending theory of Section 8 can be further reduced to an elastic–perfectly plastic plate theory which corresponds to that for the classical Poisson–Kirchhoff theory. In addition to the assumptions (9.3) and (9.6), we also allow both $\delta_\alpha \to 0$, $\alpha_3 \to \infty$ and $\delta''_\alpha \to 0$, $\eta_3 \to 0$ in such a way that V_α remains finite. With these additional assumptions, the number of boundary conditions reduces from three to two, the latter being those of the classical theory.

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REFERENCES

[1] A. E. GREEN, P. M. NAGHDI and W. L. WAINWRIGHT, *Archs ration. Mech. Analysis* **20**, 287 (1965).
[2] A. E. GREEN and P. M. NAGHDI, *Archs ration. Mech. Analysis* **18**, 251 (1965).
[3] A. E. GREEN and P. M. NAGHDI, *Proc. IUTAM Symp. Irreversible Aspects Continuum Mech.*, Vienna, 1966. Springer (1968).

* According to (9.11), the plastic strain rate is directed along the normal to the yield surface in stress space. The assumption (9.11) has been made, or is implied, in most of the literature on plasticity; it can be shown that it is also a consequence of Drucker's postulate [8]. However, we emphasize the special nature of the assumption (9.11) and point out that the preceding theory is independent of Drucker's postulate.

[4] A. E. GREEN and P. M. NAGHDI, *Q. Jl Mech. appl. Math.* **18**, 257 (1965).
[5] B. D. COLEMAN and W. NOLL, *Archs ration. Mech. Analysis* **13**, 167 (1963).
[6] A. E. GREEN and P. M. NAGHDI, *Int. J. Solids Struct.* **4**, 585 (1968).
[7] A. E. GREEN and P. M. NAGHDI, *Proc. Camb. phil. Soc. math. phys. Sci.* **63**, 537 (1967); **63**, 922 (1967).
[8] D. C. DRUCKER, *Proc. 1st U.S. natn. Congr. appl. Mech.* p. 487 (1951).

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Абстракт—Работа занимается определяющими уравнениями и относительными термо-динамическими ограничениями для упругопластической поверхности Коссера. Главные направления развития прцводится с учетом нелинейной теории. Определяются разные специльные случаи, заключающие также макце из инфинтезимальными деформациями. Детальное внимание обращается на линейную теорию для изотропной, упруго-идеально пластической поверхности (или пластинки) Коссера.